**Beginning Algebra   
Made Useful**

**Chapter 1**

Charlene E. Beckmann, Ph.D.

Grand Valley State University

© 2020

# Table of Contents

[Chapter 1 Getting Ready to Learn Mathematics 3](#_Toc536022332)

[1.1 Learning How to Learn Mathematics 3](#_Toc536022333)

[How to Learn Math for Students 3](#_Toc536022334)

[Good Questions to Ask Yourself 6](#_Toc536022335)

[Other Helpful Practices 7](#_Toc536022336)

[Mathematical Practices 8](#_Toc536022337)

[1.1 Homework 9](#_Toc536022338)

[1.2 Thinking Together and Mathematical Mindset 9](#_Toc536022339)

[Video: Mindset 9](#_Toc536022340)

[Activity: Good Group Work (Also from Week 1 of Inspirational Math) 10](#_Toc536022341)

[Activity: Mindset Survey 10](#_Toc536022342)

[Activity: Making Introductions 12](#_Toc536022343)

[1.2 Homework 12](#_Toc536022344)

[1.3 Connecting Mathematics Through Representations 13](#_Toc536022345)

[Video: Brain Crossing 13](#_Toc536022346)

[Pattern Talk: Dot Pattern 13](#_Toc536022347)

[Activity: Growing Pattern of Dots 13](#_Toc536022348)

[Activity: Graphing a Growing Pattern of Dots 15](#_Toc536022349)

[1.3 Homework 17](#_Toc536022350)

[1.4 Flexibility with Numbers and Order of Operations 18](#_Toc536022351)

[Activity: Three Threes 21](#_Toc536022352)

[Activity: Four Fours 22](#_Toc536022353)

[1.4 Homework 24](#_Toc536022354)

[1.5 Four Fours Problem, Extended 26](#_Toc536022355)

[Activity: Organizing Your Work 26](#_Toc536022356)

[1.5 Homework 27](#_Toc536022357)

[Activity: Summarizing Your Work on the Four Numbers Problem. 29](#_Toc536022358)

[1.6 Generalizations and Variables: Patterns in the 100s Chart 31](#_Toc536022359)

[Video: Brains Grow and Change 31](#_Toc536022360)

[NumberTalk: 43 – 28 31](#_Toc536022361)

[Activity: Analyzing the 100s Chart 31](#_Toc536022362)

[1.6 Homework 33](#_Toc536022363)

[1.7. More with Patterns and Variables: Rectangular Patterns in the 100s Chart 36](#_Toc536022364)

[Number Talk: 93 – 27 36](#_Toc536022365)

[Activity: Rectangle Patterns in the 100s Chart 36](#_Toc536022366)

[1.7 Homework 38](#_Toc536022367)

[1.8 Patterns and Variables: Justifying Patterns with Algebra 38](#_Toc536022368)

[Number Talk: 96 + 29 38](#_Toc536022369)

[Activity: Sharing Rectangle Patterns in the 100s Chart 38](#_Toc536022370)

[1.8 Homework 38](#_Toc536022371)

[1.9 Algebra and Number Puzzles 40](#_Toc536022372)

[Activity: Number Puzzles and Algebra 40](#_Toc536022373)

[Activity: Translating Magic Number Puzzles to Algebra to Justify Them 41](#_Toc536022374)

[1.9 Homework 42](#_Toc536022375)

# Chapter 1 Getting Ready to Learn Mathematics

## 1.1 Learning How to Learn Mathematics

Mathematics is a creative endeavor. It is full of “aha” moments. Has that been your experience with mathematics? If so, think of a few “aha” moments you’ve had. Write them down. If not, you’re about to embark on real learning of mathematics, learning that does not involve mind-numbing memorization of rules for combining symbols and solving equations.

Math makes sense. Learning mathematics requires making sense by making connections between familiar ideas you encounter in your daily life. You might find that the mathematics you encounter in this course relates to mathematical concepts from your past experiences. Some of it might seem entirely new. Whatever you think of mathematics as you begin this course, it is important that you keep an open mind, bring your creativity into your learning, and try to connect what you are learning to what you already know outside of mathematics.

There are several tools available to help you learn mathematics. In Lesson 1.1, we explore some of them.

### How to Learn Math for Students

Stanford University offers free classes online (<https://online.stanford.edu/courses/gse-yeduc115-s-how-learn-math-students>). A useful and short course entitled, *How to Learn Math for* Students, offers several short videos and brief exercises to help you build your confidence in learning mathematics. *How to Learn Math for Students* will help you learn how your brain works, how your mindset regarding mathematics helps or hinders your learning, how speed is not related to understanding in mathematics, and other useful information regarding how you learn mathematics. As part of the *Beginning Algebra Made Useful* course, you will also be completing much of the Stanford course, *How to Learn Math for Students.* Sign up for the course (it’s free!!!).

Throughout the first part of the semester, you will be assigned problems that require you to watch a video in the Stanford course, *How to Learn Math for Students*, then reflect on what you learned. The videos are short, most of them less than 3 minutes in length. After each video, you will find short quizzes and questions to answer along the way. Answer the quizzes and questions as you go. By providing a certificate of completion for the Stanford course*, How to Learn Math for Students*, you can earn privileges determined by your professor.

**Directions**: Your responses to each of the following problems will be collected on a due date provided in class. Answer each question in writing using complete English sentences. Type responses if your handwriting is difficult to read. Show all of the work you do to answer mathematical questions. Include illustrations.

1. Lesson 1: Knocking Down Myths about Math.

a. Watch the second video in Lesson 1, *How do People Feel about Math?* (3:38 minutes). Write at least 2 sentences to answer the following question: How do you feel about math?

b. Watch the third video in Lesson 1, *Math Myths and the Brain* (3:23 minutes). Answer in writing: Why do you think people often don’t like math?

c. Watch the fourth video in Lesson 1. How does the fourth video, *Brain Growth* (2:04 minutes), make you feel about your ability to learn mathematics?

d. Watch the last video in Lesson 1, *Smashing Stereotypes* (1:57 minutes). What toys and games did you experience on a consistent basis as you were growing up? Would you consider any of them to be mathematical? If so, which ones? In your opinion, what made these toys mathematical?

2. Lesson 2: Math and Mindset.

a. Watch the first Math and Mindset video in Lesson 2: *Mindset* (3:23 minutes).

b. In what areas (subjects in school, sports, etc.) do you have a growth mindset?

c. Do you tend to have a growth mindset or a fixed mindset when it comes to learning mathematics? State your evidence.

d. What messages do you give yourself that help you develop or keep a growth mindset in mathematics?

e. Watch the video, *Messages about Math* (1:44 minutes). Some students were praised for being smart. Which problem did they choose? Why do you think that students who are praised for being smart chose the problem they did?

f. Watch the video, *Messages about You* (2:29 minutes).

g. How do the videos in Lesson 2 make you feel about your ability to learn mathematics?

h. What three messages from these videos do you want to keep in mind for this and other courses you will take in college?

3. Lesson 3: Mistakes and Speed. Watch the first three videos for Lesson 3: *Mistakes: New Evidence* (1:42 minutes), *Mistakes and Success in Life* (1:29 minutes), and *Math and Speed* (3:49 minutes).

a. Do you see mistakes in math as a sign of failure or a sign of persistence?

b. Write a paragraph about what you’ve learned through the videos in Lesson 3.

c. Open problems are problems for which there are multiple approaches to find a solution or there are multiple solutions. Most of the problems we’ve been working on in class are considered open problems. How do they help you learn? What about them, if anything, causes you anxiety?

4. Lesson 4: Number Flexibility, Mathematical Reasoning, and Connections.

a. Watch the first video, *Number Flexibility* (1:24 minutes), then simplify 18 x 5 using a mental strategy. Write down how you thought about the problem as you solved it.

b. Watch the second video, *18 x 5* (5:27 minutes). In writing, answer the question posed at the end of the video. Match each illustration with the solution process that fits it. Include drawings in your submission.

c. Watch the third video, *12 x 15* (1:18 minutes). Solve the problem stated without using the usual algorithm. Try to solve the problem mentally first using friendly numbers. Then try to draw a picture that goes with the way you solved the problem.

d. Solve the problem in problem 4c another way, and again without paper and pencil. Draw a picture that illustrates the way you solved the problem this time.

5. Continuing with Lesson 4: Number Flexibility, Mathematical Reasoning, and Connections.

a. Watch the fourth video, *Talking About Math* (1:41 minutes). Why did the students who worked on math together do so well in their mathematics classes?

b. Watch the fifth video, *Reasoning* (1:32 minutes). What is reasoning? Why is it important?

c. Watch the sixth video, *Mathematical Connections* (9:10 minutes; though this video is long, listen all the way through). Several connections were drawn in the video. Choose 3 of the examples and explain how they are connected.

d. Watch the seventh video, *PISA Results* (0:41 minutes). Write about some of the connections you have made in mathematics in this class. Why are they important to your learning?

6. Lesson 5: Number Patterns and Representations

a. Watch the first video in Lesson 5, *Making Sense and Intuition* (4:04 minutes). Solve the problem posed immediately after this video. Show how you solved the problem. Do not go on to the next video until you find a solution you think works.

b. Watch the second video in Lesson 5, *Drawing and Representing* (5:11 minutes). How did the video solution compare with your solution in problem 6a? How did the video help you think about the problem differently?

c. What does this video tell you about *How to Learn Math*?

d. What important information do you want to remember from these two videos?

7. Continuing with Lesson 5: Number Patterns and Representations.

a. Watch the third video in Lesson 5, *Looking for Big Ideas* (5:27 minutes). Write down and answer the quiz question after the video. Explain your response. Use what you learned in the video to determine which is bigger, or . Include illustrations in your solution.

b. Watch the fourth video, *Fractions: Same Size Pieces* (1:11 minutes). Take the quiz following the fourth video. Write down the problem, show your response, and explain your answer.

c. What fraction represents the area of the white triangle in the figure? Explain your answer.

d. Watch the fifth video, *Fractions: Big Ideas* (2:01 minutes), more than once if needed.

e. From the videos in problems 4b and 4d, what big ideas about fractions will help you make sense of them? List at least two.

f. Watch the sixth video in Lesson 5, *Ideas vs. Memorization* (0:53 minutes).

i. Think about the mathematical content in our class. What big ideas have you learned so far? What makes them big ideas?

ii. What important strategies have you learned? How might these strategies help you in your future mathematics learning?

g. What has surprised you about your learning of mathematics?

h. What additional mathematics would you like to learn?

8. Think about your work with mathematics in and out of class so far this semester.

a. As you work through activities and problems in and out of class, what messages are you giving yourself? Have you been persistent? Have you told yourself you can do this?

b. The problems we work on in class and for homework often require persistence. What do you do when you struggle with a problem?

c. Have any of your messages to self been negative? If so, based on the videos from *How to Learn Mathematics for Students*, what messages help you grow your brain?

9. Answer the following questions. Be specific. Provide examples.

a. Why is it important to your mathematical learning to talk about math with others, to make connections, and to use reasoning?

b. How might you use these ways of learning to help you in other subjects?

c. How might you use these ways of learning to help you in your daily life?

10. Think about your views of your ability to learn math.

a. Have any of your views about your ability to learn math changed this semester? Explain.

b. Watch Video 9 in Lesson 6, *Summary and Reflections* (2:01 minutes). What ideas from *How to Learn Math for Students* stand out for you.

c. Take the survey immediately following Video 9. List the top 5 ideas from the course that stand out for you.

d. Write a reflection, at least 2 paragraphs long, about what you have learned about your ability to learn mathematics.

11. Earn a certificate for completing the Stanford course, *How to Learn Math for Students*. You may earn privileges as designated by your professor for completing the course and turning in a printed copy of the certificate of completion.

### Good Questions to Ask Yourself

When working on mathematics problems, you are in good company if you find that you are often or occasionally stuck. There is always hope. The questions that follow have been used many times to help students find a way out of a temporary dilemma when solving a problem. First, try to identify what’s causing your dilemma.

* Do you understand what the problem is asking? If not, you are probably stuck at Stage 1.
* Do you understand the problem but don’t know what to do now? You are likely stuck at Stage 2.
* Have you started to solve the problem and gathered some information but don’t know what to do with what you’ve found? You might be stuck at Stage 3.
* Have you finished the problem? Stage 4 questions can help you organize what you’ve learned through solving the problem into what you already know and help you move new learnings into your long-term memory.

Goldin (1998) suggests a hierarchy of four stages of questioning to guide yourself or a peer to solve a problem:

Four Stages of Questioning

##### Stage 1: Understanding the Problem

Read the problem and take time to make some initial progress on it. Ask yourself:

* What is the problem asking me to do? Write down ideas in your own words.
* What do I know?
* What am I trying to find?

##### Stage 2: Devising a Plan

If you understand the problem and have trouble starting to solve it, think about general problem-solving strategies that might help. Ask yourself:

* Would drawing a picture help?
* Would a table of values help to solve this problem?
* Would any materials help to model the problem? (Get the materials and use them.)
* Is there a similar problem I’ve done before that might help me get started?
* What do I know must happen in the problem?
* What do I know cannot happen in the problem?

##### Stage 3: Carrying Out the Plan

If you get stuck once you are partway through solving the problem, think about more specific problem-solving strategies. Ask yourself:

* What patterns do I see in the table of values or drawings?
* What rule can be used to describe the relationship in the table of values I created?
* What information does the graph give me?

##### Stage 4: Looking Back

Once you solve a problem, think back about what you did. Think about your thinking by asking yourself:

* How do I know my solution is correct? How can I convince someone else?
* How did I think about the problem?
* Can I solve the problem in another way?
* How does this problem connect with other problems I have solved?

Your goal is to help yourself explain the problem and its solution. This stage helps you mentally file your thoughts about the problem in a category with other similar problems, organizing your new learning into your long-term memory.

Table based on Goldin, Gerald A. “Observing Mathematical Problem Solving through Task-Based Interviews.” In Anne R. Teppo (Ed.), *Qualitative Research Methods in Mathematics Education* (pp. 40-62). Reston, VA: National Council of Teachers of Mathematics, 1998.

### Other Helpful Practices

To best learn mathematics, you will find that each of the following strategies are helpful if you make them a habit:

* Read carefully. It might be necessary to read problems or narration several times in order to understand more fully what each intends. Students who do not read carefully always make mistakes that can be remedied by rereading more carefully.
* Organize your work. Use tables and graphs to help you make sense of information you find when solving a problem. Don’t underestimate the power of trying a few numbers to see what happens. Always keep track of what you try. Do not erase or throw away attempts that don’t work. Save them and compare them to an approach that did finally work. Ask yourself these questions: What was I thinking initially? What do I now know? Was I far off at the beginning? If not, what was I thinking that could have led me to a correct solution? If so, what was I thinking that was helpful? What was getting in the way of my understanding?
* Check your solutions to see that they make sense. If a context is given, translate your solutions into the context. Check to make sure any numbers you’re getting from equations, tables, or graphs make sense in the context. Make sure graphs, tables, and equations give corresponding information.
* Never give up! It’s OK to take a break from a problem that’s frustrating you. Return to it after you’ve had a chance to relieve your frustration. You can learn mathematics. You can solve any problem facing you, AS LONG AS YOU DON’T GIVE UP!
* Study at least a little most days. After each class, take 15 minutes to write down what you learned. Be specific. For example, don’t state, “I learned about linear functions.” Instead, state specifically what you learned, “I learned that the slope of a linear function when shown in a graph is the value of *m* in the equation, *y* = *mx* + *b*. This makes sense, because slope is how much *y* changes when *x* changes by 1 so in a graph, the *y*-value ofa linear function changes by *m* every time *x* changes by 1.”
* Do your assigned homework. Homework provides you the opportunity to grow your brain outside of class. It also prepares you to think about problems that build on your homework when you are in class. Making mistakes is an important part of learning, so grading your early thoughts about a topic is counterproductive. DOING homework, however, grows your brain, allows you to get more out of the work you’re doing in class, and helps you prepare for your future with technical information. Not doing homework is detrimental to you, and to your current and future work!
* Commit to your education. Participate fully in your group and class and make every effort to complete your homework so that you’re prepared to participate with your group.

### Mathematical Practices

Several practices have been identified that can help you make sense of problems that arise in your daily work and in technical situations. Which practices do you use to learn mathematics and other content fields? When you work on mathematics, use this page to help you choose strategies that might help you learn. Revisit this page often, especially when you get stuck, for suggestions of other strategies you might try as you’re trying to solve problems. (Resource: <http://www.corestandards.org/Math/Practice/>)



### 1.1 Homework

Complete the following homework before the next class period:

1. Go back to the first section of Lesson 1.1, *How to Learn Math for Students*. Complete problem 1 on pp. 3 and 4. You will hand in your written reflections for problem 1 in class.

2. Read the sections above, *Good Questions to Ask Yourself*, *Other Helpful Practices*, and *Mathematical Practices*. Remember these sections as you work your way through the activities in this course. Refer to each section as needed. Write a one sentence description of each section to help you remember when to consult each of these sections.

## 1.2 Thinking Together and Mathematical Mindset

### Video: Mindset

The message of “Mindset” is that everyone can learn mathematics. The video includes important brain and mindset evidence to support this message. Watch the video. Write a brief reflection, then discuss the video in small groups. This video is from Week 1 of Inspirational Math, Day 1: <https://www.youcubed.org/resources/mindset-video/>.

### Activity: Good Group Work (Also from Week 1 of Inspirational Math)

"If you want to go fast, go alone, if you want to go far, go together." African Proverb

Before you work on math together, complete this activity. Use the cooperative learning strategy, *Roundtable*, to generate ideas.

#### Roundtable:

* Each team uses a single sheet of paper.
* The team member closest to the classroom door writes down an idea, states it out loud, and passes the paper to the left.
* Other group members may ask for clarification but may not critique the idea.
* Continue until time is called.

This activity will help improve group interactions. Teachers who have tried this activity have been pleased by students’ thoughtful responses and found students’ thoughts and words helpful in creating a positive and supportive environment. Use *Roundtable* to:

* Reflect on things you like people to say or do in a group when you are working together.

After you have thought of and written down a few of the ideas, use *Roundtable* to:

* Reflect on things you don’t like people to say or do when you are working in a group.

Some important ideas students generally share include: Let everyone share their ideas, share different ways to solve a problem, listen when others speak, don’t give away the answer, don’t rush through the work, and don’t ignore other people’s ideas.

When the class has finished brainstorming, share ideas on a “What we like” poster with each group contributing at least one idea. Do the same for a “What we don’t like” poster. Post the final posters in class as the agreed upon classroom norms; refer back to them throughout the course, as needed. Don’t include negative comments such as, “I don’t like waiting for slow people,” instead discuss the issue.

### Activity: Mindset Survey

Document your current feelings about how you approach mathematics through the following:

1. Look at the *Mathematical* *Mindsets* figure on page 11. Follow the directions to get a baseline of your attitudes concerning your ability to learn mathematics.

2. a. The boxes in the first column of the *Mathematical Mindsets* figure describe a person with a Fixed Mindset. The boxes in the second column describe a person with a Growth Mindset. At the present time, do you tend to have a Fixed Mindset or a Growth Mindset in terms of learning mathematics? Explain why you think so.

b. Having a Growth Mindset helps you overcome obstacles as you work to learn mathematics. Having a Fixed Mindset, on the other hand, can prevent you from learning mathematics. It is important to work to overcome a Fixed Mindset so that it doesn’t hold you back from learning. If you tend to have a Fixed Mindset, what ideas do you have to help you transition from a Fixed to a Growth Mindset?

3. Keep your responses to problems 1 and 2. Revisit them often. Each time, ask yourself:

* Have any of my beliefs changed?
* Do I believe in myself and my ability to learn mathematics?

If the answer to either question is no, ask yourself,

* What messages am I giving myself that keep me from believing in myself?
* What messages would be more helpful?



### Activity: Making Introductions

In this activity, in turn, you will introduce yourself using alliteration. You will also introduce everyone who was introduced before you. Suppose the first 3 persons are Char, Ben, and Melanie. Char starts by saying: “I’m Char. I like shopping in Chicago.” Ben says, “I’m Ben. I like baseball. This is Char; she likes shopping in Chicago.” Melanie might say, “I’m Melanie, I like making meals. This is Ben; he likes baseball. This is Char; she likes shopping in Chicago.” Introductions continue around the room with successive students introducing everyone who came before them. If there are 30 students or more in class, the first half of the students introduce themselves using this scheme, then the second half does the same starting over with the first person of the second group.

Once introductions have been made, complete the following:

* Create a table to show the number of introductions made by each of the first 5 people and also the total number of introductions made so far.
* Look for and share any patterns you see in the table.
* Determine the total number of introductions that have been made after 10 people, 15 people, and 20 people.
* Is there a way to find the number of introductions made by the whole class? How? What is the number? How do you know?

### 1.2 Homework

Complete the following homework before the next class period:

1. Complete problem 2, on page 4 in Lesson 1.1, *How to Learn Math for Students*. You will hand in your written reflections for problem 2 in class.

2. Answer the bulleted questions just above this homework assignment. Be ready to share your responses with the class.

3. Take the Mindset Survey online at: <https://www.mindsetworks.com/assess/>. Read the recommendations provided once you take the survey. Print out a copy of these recommendations or store the results on your computer. Consult the results any time you feel frustrated with your work in this course.

## 1.3 Connecting Mathematics Through Representations

Much of mathematics involves the study of patterns. Making sense of patterns happens through interconnections among representations. In Lesson 1.3, we begin with a visual growing pattern of dots, connect it with a pattern of numbers, and graphs that model different interpretations of the pattern. Finally, we connect the pattern to a familiar context. All of these representations help us makes sense of one or more aspects of how the pattern is growing, how to analyze it, and how to generalize it.

### Video: Brain Crossing

The video shares some important new research on the power of engaging with numbers and symbols visually, which involves brain crossing. Your brain is capable of learning through visual cues, especially if you connect them to other modes of learning. The video explains that it is helpful for all students to think visually about mathematics. “Brain Crossing” is found in Week 1 of Inspirational Math, Day 2: <https://www.youcubed.org/resource/mindset-boosting-videos/>.

### Pattern Talk: Dot Pattern

Consider the dot arrangement at right. How many dots are in the arrangement? Explain how you counted the dots. Share as many ways of counting the dots as possible. Show the different ways you counted the dots by drawing on each figure below.

Mathematics is a creative endeavor. There is always more than one way to solve a problem. If you get stuck, try to find another route!



### Activity: Growing Pattern of Dots

Now consider a pattern related to the dot arrangement above. See the student page, *Growing Pattern of Dots*.

Spend at least 5 minutes thinking about the problem on your own.

When everyone in your group is ready, share your solutions to *Growing Pattern of Dots* with each other. Try to resolve any differences you have in your understandings. If you and your group members are still uncertain about something after your discussion, write a question or two to ask the full class.

Growing Pattern of Dots

Consider the pattern of dot arrangements below.



1. How are these dots related to the dots you counted in the Pattern Talk?

2. a. How is each arrangement of dots related to the arrangement in the step before it?

b. How is each arrangement of dots related to the arrangement in the step after it?

3. Is this pattern of dots related to *Making Introductions*? How?

4. Consider the strategies you and classmates used to count the arrangement of dots in the Pattern Talk. Choose one strategy that works to count the dots in each of the arrangements.

a. Describe the strategy.

b. Use the strategy, show how you used it for each step, and record the number of dots in each arrangement in the bottom row of the table above.

5. How many dots will there be in step 10? How do you know?

Share your responses with the full class:

* Which strategy did you use to count the dots in each of these arrangements? Show how you used the same strategy to count the dots in Steps 4 and 5.
* Does the strategy you used to count each dot arrangement help you determine the number of introductions that were made in our class? Explain.
* What questions do you still have? (Resolve these as a class.)

### Activity: Graphing a Growing Pattern of Dots

Consider the graphs on the student page, *Graphing a Growing Pattern of Dots*. For each graph, determine if the graph is correct. Explain why or why not.

Graphing a Growing Pattern of Dots

1. Consider the data table you created for the *Growing Pattern of Dots.* The data is plotted in the following graphs. For each graph, determine:

a. Is the graph a correct representation of the data?

b. Why or why not?



2. What are some important characteristics of an accurate graph? List of at least 5 items.

This collection of activities — *Making Introductions, Counting Dots, Growing Pattern of Dots,* and *Graphing a Growing Pattern of Dots —* shows how seemingly unrelated contexts and representations can have the same underlying mathematical structure. We see this a lot in mathematics. Making these connections requires Brain Crossing (recall the video you watched today). Remember that Brain Crossing helps you with recall and grows your brain!

When you work on a new problem, try to connect the work you’re doing to problems you’ve worked on in the past. Connecting related mathematics problems helps reduce your reliance on memorized procedures. It also helps move your working (short-term) memory into long-term memory so that it can be retrieved to help you make sense of new problems and concepts.

### 1.3 Homework

Complete the following homework before the next class period:

1. Go back to the first section of Lesson 1.1, *How to Learn Math for Students*. Complete problem 3 on page 4. You will hand in your written reflections for problem 3 in class.

2. Revisit each of the related activities *Making Introductions, Counting Dots, Growing Pattern of Dots,* and *Graphing a Growing Pattern of Dots*. Write a paragraph or two that shows your understanding of how each of these activities is related.

3. Critique the graph at right. If the graph is accurate, say how you know. If the graph is incorrect, indicate all the ways that it needs to be fixed, label the axes correctly and redraw the graph.

4. The same growing number pattern is shown twice in the figure below. Illustrate on the figures and explain in words how you are counting each set of dots in the pattern.

a. Count the number of dots in the first set of figures using the same method to count all three arrangements of dots.

b. Count the number of dots in the second set of figures using a different method from the method you used in problem 4a. Use this second method to count all three arrangements of dots.

c. What number of dots will be in the next figure? How do you know? Include a drawing of the figure in your explanation.

5. Consider each of the stories and expressions on the student page, *Interpreting Expressions*. What do you notice about the expressions? Which ones name the same number? Why?

Interpreting Expressions

1. Decide which numeric expression best fits each story. Be ready to defend your choices.

2. Decide which numeric expressions, if any, represent the same value.

|  |  |
| --- | --- |
| I am making apple pie and apple turnovers. One recipe calls for 2 apples. The other recipe calls for 3 apples. I’m making 5 recipes of each dessert. How many apples do I need? | 2 + (3 x 5) |
| I have 2 one-dollar bills and 3 five-dollar bills. How much money do I have? | (2 + 3) x 5 |
| Mom lets me keep any money I find when I clean the house. She also quadruples any money I find to pay me for my work. I found $2 in the couch and $3 in my dad’s jacket. How much money will I have after Mom pays me? | 2 + 3 x 5 |
| I need 2 cups of almond flour to make a batch of pancakes and 3 cups of almond flour to make a cake. I’m going to make 5 cakes and 1 batch of pancakes. How much flour will I need? | 2 + 3 + (2 + 3) x 4 |

6. Read the introduction to Lesson *1.4 Flexibility with Numbers and Order of Operations*. In writing, answer each question in the reading.

## 1.4 Flexibility with Numbers and Order of Operations

In Lesson 1.4, you will use your creativity and play with mathematical operations using the Order of Operations. The Order of Operations is an agreed upon order in which operations must be completed when simplifying a numeric or an algebraic expression. When presented as a list of rules, the Order of Operations can seem arbitrary. However, there are mathematical reasons underlying the order in which operations are completed.

The Order of Operations follows the historical introduction of operations. Addition happened first. Subtraction is the operation that undoes addition. Multiplication naturally occurred as people needed to determine multiple sums of the same number; multiplication is repeated addition. The operation that undoes multiplication is division; division can also be thought of as repeated subtraction. Exponentiation occurred even later signifying repeated multiplication. Taking a whole number root, the inverse operation of raising a number to the same whole number exponent, can also be thought of as repeated division. When you encounter an expression to be simplified, consider the historical introduction of arithmetic operations and perform the most complex operation first, reducing each subsequent expression in the opposite order that the operations arose historically. Of course, if parentheses are included to alter the order of operations further, expressions inside them must be simplified first!

Let’s consider some examples. To simplify the expression, 3 + 2 x 7, recall that the multiplication sign represents repeated addition. An equivalent expression written entirely in terms of addition is 3 + 7 + 7 = 17. Try this with your calculator; do you get this result? Some students want to add 3 and 2 then multiply the result by 7 because the expression is written that way. If you add first to get 5 then multiply by 7, you get 35. This is not equivalent to the original expression. Notice that 35 = (3 + 2) x 7. In the original expression, only 2 is multiplied by 7. This example helps us understand why multiplication is completed before addition when simplifying 3 + 2 x 7, and in general.

Now consider the expression, 5 x 32. As indicated above, terms with exponents represent repeated multiplication. The expression, 5 x 32, can be rewritten as 5 x 3 x 3. To multiply 5 x 3 then square the result changes the meaning of the original expression to (5 x 3)2. Try both versions on your calculator. Because no parentheses are included in the original expression,   
5 x 32 ≠ (5 x 3)2.

Try to simplify this expression to see why the Order of Operations is helpful. Change all exponentiation to repeated multiplication then change all multiplication to repeated addition before simplifying.

5 + 2 x 32

Look at the simplification below. Explain how each expression arises from the one before it:

5 + 2 x 32 = 5 + 2 x 3 x 3

= 5 + (3 x 3) + (3 x 3)

= 5 + (3 + 3 + 3) + (3 + 3 + 3)

= 5 + 9 + 9

= 23

You can see from this simplification and your explanation that simplifying an expression by replacing exponentiation with repeated multiplication and multiplication with repeated addition will quickly get cumbersome.

The rules for simplifying expressions using the Order of Operations serve as a reminder. To recall which operation to complete first it is helpful to think about what each operation represents. There are cautions, however. Sometimes the Order of Operations can lead us to believe that the order in which we perform operations is rigid. We know that expressions like 85 + 27 + 15 and 25 x 11 x 4 can be simplified more easily if we rearrange the terms. We see that 85 + 15 = 100 so the sum 85 + 27 + 15 = 85 + 15 + 27 = 100 + 27 = 127. We also see that   
25 x 4 = 100 so 25 x 11 x 4 = 25 x 4 x 11 = 100 x 11 = 1100. Addition and multiplication are commutative (the order can be changed without changing the result). We also know that we can group numbers differently when they are all combined with addition or with multiplication; this is the associative property: 14 + (16 + 18) = (14 + 16) + 18, 15 x (6 x 17) = (15 x 6) x 17. Both of these properties allow us to group and rearrange numbers combined with one of the operations + or x (not both together, though) to put friendlier, easier to combine, numbers together to aid in simplification.

We would expect that the Order of Operations is a universal convention, agreed upon and used everywhere in the world in the same way. This is not the case, as it happens. Consider the case of multiplication and division. The Order of Operations convention used in USA textbooks tells us to complete multiplication and division in the order they appear in an expression, left to right. In Kenya, students are told to complete division first, in the order division arises, then complete multiplication. Simplify the following expressions with each convention. What do you notice?

15 x 21 ÷ 7 x 5 9 x 24 ÷ 6 ÷ 3 x 12 35 ÷ 7 x 18 ÷ 2

What do you think is happening here? Can you explain why the result is the same regardless of which version of the Order of Operations you use? Think about how multiplication and division are related then rewrite each expression to use only multiplication. Now what do you notice?

Which Order of Operations convention do you think is easier and more reliable to implement? Use that one! The following reminders of Order of Operations might be helpful. Use whichever one makes more sense to you. Notice that the order you simplify expressions, in both conventions, is the opposite order that each operation was introduced historically; more complex operations are simplified first.

Resource: Bay-Williams, Jennifer M. and Sherri L. Martinie. *Order of Operations, The Myth and the Math*. Teaching Children Mathematics,Vol. 22, No. 1, August 2015.

### Activity: Three Threes

Consider theThree Threes Clock Face in the student page below. Verify that each number on the clock face is represented correctly using three threes. Think about how you are using the Order of Operations to check each expression. Use your creativity to find as many expressions for each number as you can.

Three Threes Clock Face

Study the clock face.

1. Verify that each expression is correct. Show your work. (As they arise, write any questions you have about any of the expressions.)

2. Find other expressions for some of the numbers 1 through 12 using three threes. Show your work.

Share your thoughts on how each number on the clock was represented by three threes. Also share alternative ways you represented some of the numbers using three threes.

### Activity: Four Fours

Now that you’ve had a chance to verify and play with expressions for the numbers 1 through 12 created using three threes, use your creativity to find expressions for other numbers using four fours. The *Four Fours Problem* provides you additional opportunities to think about the order of operations and the mathematical operations you have studied in previous courses. Study the problem presented on the *Four Fours Problem* student page.

How can you make the numbers 1 through 20, 24, 30, and 100 using 4 fours? Record all of the expressions you find for each number in the table. Work alone for at least 5 minutes before you share your results with your group. If you get stuck, consult the *Three Threes Clock Face* for inspiration.

Four Fours Problem

Use the number 4 exactly four times to write expressions equivalent to the numbers from 0 through 20. You may use any of the mathematical symbols below. A few examples are provided.

|  |  |
| --- | --- |
| * Arithmetic operations: +, –, x, ÷ | * Square root: e.g. |
| * Parentheses: ( ) | * Factorial: ! (4! = (4)(3)(2)(1) = 24) |
| * Decimal points or percents: e.g. 4.4, .44, 44% | * Concatenation: e.g. 44, 444 |
| * Exponentiation: ^ (to the power of), e.g. (4 + 4)4 | * Repeating decimal: |

1. Complete the table. Find at least one expression using 4 fours for each number. Use the Order of Operations; include grouping symbols as needed. Check your responses using a graphing calculator. Some examples for *n* = 4 are provided. For the numbers whose solutions are shown in the table, find at least one more expression that works.

|  |  |  |  |
| --- | --- | --- | --- |
| **Number** | **Four Fours Equation** | **Number** | **Four Fours Equation** |
| 0 |  | 12 |  |
| 1 |  | 13 |  |
| 2 |  | 14 |  |
| 3 |  | 15 |  |
| 4 |  | 16 |  |
| 5 |  | 17 |  |
| 6 |  | 18 |  |
| 7 |  | 19 |  |
| 8 |  | 20 |  |
| 9 |  | 24 |  |
| 10 |  | 30 |  |
| 11 |  | 100 |  |

2. What strategies did you use to find solutions?

3. Find more than one solution for at least 3 of the numbers. Record them in the table.

4. Which of your solutions are not dependent on the number 4? For example is also true when each 4 is replaced with 2, 6, or 9. Highlight these solutions in your table.

Share the expressions you found using Four Fours with your group. Compare your work with your group members. Do you agree that each expression is correct? Circle any expressions you question; put a checkmark next to expressions with which you agree. Resolve any differences of opinion. When asked to do so, share some of your group’s expressions with the class.

### 1.4 Homework

Complete the following homework before the next class period:

1. Go back to the first section of Lesson 1.1, *How to Learn Math for Students*. Complete problem 4 on page 4. You will hand in your written reflections for problem 4 in class.

2. Determine the value of each expression without the use of an electronic tool. Show your work one step at a time.

a. b.

3. While working on the *Four Fours Problem*, students solved the following expression in these four ways.

a. Only one solution can be correct. Which one is it? How do you know?

|  |  |  |  |
| --- | --- | --- | --- |
| = 4 + 4 / 2 x 2  = 4 + 4 / 4  = 4 + 1  = 5 | = 4 + 4 / 2 x 2  = 8 / 4  = 2 | = 4 + 4 / 2 x 2  = 4 + 2 x 2  = 4 + 4  = 8 | = 4 + 4 / 2 x 2  = 4 + 2 x 2  = 6 x 2  = 12 |

b. For the solutions above that are incorrect, identify the error(s) the student made.

c. As you work on the *Four Fours Problem* check the expressions you find to make sure they work for the numbers you think they do.

4. Solve each of the following. Show how you know your work is correct.

a. Insert parentheses so this equation is true: .

b. Insert parentheses so this equation is true: = 4.

c. Revisit problem 2a. With no parentheses in the left side of the equation, what would the simplification of the expression be? Write the expression and show the simplification.

5. Look back at your work on the *Four Fours Problem*. Indicate where you used the mathematical properties and operations listed:

* Order of Operations
* Additive identity (adding zero, 4 +(4 – 4) = 4 + 0 = 4; )
* Multiplicative identity (multiplying by 1= )
* Fractions as division,
* Square root:
* Exponentiation:
* Factorials: 4! = 4 • 3 • 2 • 1 = 24
* Concatenation: 44, 444
* Decimals: 4.4, .4,

6. What strategies have you used to find different expressions? Describe at least three.

7. Which of the expressions you found will give you the same result if you use four copies of another number such as 1, 2, 3, …? For example, ; notice that this relationship also works if you replace all 4 fours with four of any of these single digit numbers: 1, 2, 3, 5, 6, 7, 8, or 9.

8. Two students’ work on the same math problem is shown below. Assess their work.

a. What error(s), if any, did each student make?

|  |  |
| --- | --- |
| Student 1: | Student 2: |

b. If neither solution is correct, find the correct solution and show how to find it step-by-step.

## 1.5 Four Fours Problem, Extended

### Activity: Organizing Your Work

In mathematics, it is often helpful to create an organized list. The student page, *Ways to Use 1, 2, or 3 Fours to Make Other Numbers*, helps you organize expressions you have found using fewer than 4 fours.

Ways to Use 1, 2, or 3 Fours to Make Other Numbers

1. As you worked on the Four Fours Problem, you likely found many ways to combine fewer than 4 fours. Use this table to organize what you’ve found. Find other ways to combine 1, 2, or 3 fours to make other numbers. For each column, order the combinations you find least to greatest.

|  |  |  |
| --- | --- | --- |
| **Using 1 Four** | **Using 2 Fours** | **Using 3 Fours** |
|  |  |  |
|  |  |  |
|  |  |  |
|  |  |  |
|  |  |  |
|  |  |  |
|  |  |  |
|  |  |  |
|  |  |  |
|  |  |  |
|  |  |  |
|  |  |  |
|  |  |  |
|  |  |  |

2. Combine some of the expressions you found in the table above to make numbers using four fours. How can this organized list help you find solutions for some of the numbers in the *Four Fours Problem*.

3. What is the smallest number greater than zero you can make using four fours? What strategies did you use to find this number?

4. What is the largest number you can make using four fours? What strategies did you use to find this number?

Share your solutions with your group. Verify each other’s work. Get all possible answers on the board for the numbers 1 through 20, 24, 30, and 100. If there are any numbers for which solutions haven’t been found, work on these together.

Once remaining questions about the *Four Fours Problem* are resolved, if time remains, work on the *Four Numbers Problem* in your small group.

### 1.5 Homework

1. Go back to the first section of Lesson 1.1, *How to Learn Math for Students*. Complete problem 5 on page 5. You will hand in your written reflections for problem 5 in class.

2. The *Four Numbers Problem* extends the *Four Fours Problem*, this time using four copies of the same number. Your group will choose one of the numbers 1, 2, 3, 5, 6, 7, 8, or 9 (no group may choose the same number as another group). This is your first group assignment. Most of the work on this group assignment will occur outside of class. Exchange contact information and arrange times to get together electronically or in person to work on this assignment.  
Your group’s chosen number is \_\_\_\_\_\_\_\_\_\_.   
The completed *Four Numbers Problem* is due in class on \_\_\_\_\_\_\_\_\_\_.

3. Adapt the student page, *Ways to Use 1, 2, or 3 Fours to Make Other Numbers*, to help you solve the *Four Numbers Problem*.

4. Once you complete the *Four Numbers Problem*, you will also submit a reflection about how well your group worked together to solve the *Four Numbers Problem*. Answer the following questions to reflect on your work as a group. Turn in this reflection on the day you submit your group report for the Four Numbers Problem.

a. Did you and your group members get together outside of class or communicate outside of class to complete the work on the Four Numbers Problem?

b. Did each group member come prepared to group meetings ready to contribute individual work to the group discussion?

c. Did some group members only work on the problem when you were together? If so, characterize their contributions.

d. What steps can your group take to help your group work more productively together?

e. Are you satisfied with your group? If not, what would need to happen for you to be satisfied with your group?

f. Would you like to stay with your current group? If you want to switch groups, with whom would you like to work?

Flexibility with Numbers: Four Numbers Problem

Use the same number exactly four times to write expressions equivalent to the numbers   
0 through 20. You may use any of the mathematical symbols below. A few examples for the number4 are provided. Choose the number you will use and tell your instructor. You may choose one of the numbers: 1, 2, 3, 5, 6, 7, 8, 9. (Note that you may not use the number 4.)

|  |  |
| --- | --- |
| * Arithmetic operations: +, –, x, ÷ | * Square root: e.g. |
| * Parentheses: ( ) | * Factorial: ! (4! = (4)(3)(2)(1) = 24) |
| * Decimal points or percents: e.g. 4.4, .44, 44% | * Concatenation: e.g. 44, 444 |
| * Exponentiation: ^ (to the power of), e.g. (4+4)4 | * Repeating decimal: |

1. Complete the table writing solutions as expressions. Use the Order of Operations; include grouping symbols as needed. Check your responses using a calculator. Some examples for the number 4 are provided. Your group’s number is \_\_\_\_\_\_\_\_\_\_.

|  |  |  |  |
| --- | --- | --- | --- |
| **Number** | **Four Numbers Equation** | **Number** | **Four Numbers Equation** |
| 0 |  | 12 |  |
| 1 |  | 13 |  |
| 2 |  | 14 |  |
| 3 |  | 15 |  |
| 4 |  | 16 |  |
| 5 |  | 17 |  |
| 6 |  | 18 |  |
| 7 |  | 19 |  |
| 8 |  | 20 |  |
| 9 |  | 24 |  |
| 10 |  | 30 |  |
| 11 |  | 100 |  |

2. What strategies did you use to find solutions?

3. Find more than one solution for at least 3 of the numbers.

4. Which of your solutions are not dependent on the number you chose? Highlight these solutions in your table.

### Activity: Summarizing Your Work on the Four Numbers Problem.

Use the student page, *Reflecting on Your Group’s Work on the Four Numbers Problem*, to guide your group’s reflection on your work on the *Four Numbers Problem*. Look for similarities in how numbers were created. Keeping the Order of Operations in mind, check each other’s solutions to see if you agree. Resolve disagreements.

Write a paragraph together indicating what you individually and collectively learned about some mathematical ideas through your individual and group work on the collection of activities:

* *Three Threes Clock Face,*
* *Four Fours Problem,*
* *Ways to Use 1, 2, or 3 Fours to Make Other Numbers*, and
* *Four Numbers Problem.*

Write your name on your individual work and on the group reflection page. In your group, collect each member’s work on the *Four Numbers Problem*. Staple *Reflecting on Your Group’s Work on the Four Numbers Problem* to the top of this collection to submit.

The Four Fours activities were designed to help you:

* Have fun with mathematics,
* See how mathematics can be a puzzle to play with,
* Experience how flexible problem solving can be (There were many expressions for each number, allowing you to use your creativity to find them.),
* Persevere in mathematics when you get stuck,
* Review and remember the Order of Operations,
* Recall and use many mathematical operations (+, –, x, ÷, !, exponentiation),
* Become more familiar with your graphing calculator’s operation, and
* Experience generalization; several numbers could be expressed in the same way using any single digit integer.

One example of generalization in the Four Numbers Problem is: . This equation works in the same way for 2, 3, 6, and 7. If you replace each numeral with 0! = 1, the number 2 can be represented in this way for 0! and all integers 1 through 9. To generalize this equation, we let *n* represent any single digit non-zero integer and replace the numerals in the equation with *n*. The equation becomes indicating that as long as each *n* in this equation is replaced by the same number, the equation is always equal to 2. The letter *n* is a *variable*. It takes the place of a number, allowing us to be free of having to show the equivalence using each number as we did at the beginning of this paragraph. This is a very powerful idea.

Reflecting on Your Group’s Work on the Four Numbers Problem

1. Share your Four Numbers responses with your group. On your group member’s sheet, highlight any expressions you find particularly interesting and any expressions you are unsure of so that you may discuss them together.

a. Do you agree with each of the solutions each group member found? If not, resolve any difficulties. Do not erase any of a student’s original work. Instead, note any expressions over which group members had differences of opinion on the back of this page. Also indicate how you resolved them.

b. Look for similarities in how numbers were created. What do you notice?

2. For which numbers did you and your group members have the most numerical expressions?

a. Write the numbers and as many different responses as you can in the table below.

b. What made these responses easy to find?

|  |  |  |  |
| --- | --- | --- | --- |
| **Number** | **Four Numbers Expression** | **Number** | **Four Numbers Expression** |
|  |  |  |  |
|  |  |  |  |

3. For which numbers did you and your group members have the smallest number of numerical expressions?

a. Write the numbers and the different responses group members found in the table below.

b. What made these responses difficult to find?

|  |  |  |  |
| --- | --- | --- | --- |
| **Number** | **Four Numbers Expression** | **Number** | **Four Numbers Expression** |
|  |  |  |  |
|  |  |  |  |

4. Were there any numbers for which your group found no numerical expressions using four numbers? Write the numbers below. We will work on these together as a class.

## 1.6 Generalizations and Variables: Patterns in the 100s Chart

We saw that variables can be used to generalize expressions in the *Four Numbers Problem* such as . Variables can also be used in other ways. In this lesson, you will explore other ways that variables can be used in algebra.

### Video: Brains Grow and Change

The video explains that making sense of mathematics is necessary to grow your brain. It shares brain science related to how to strengthen or create new pathways in your brain. If somewhere in your past, you have gotten the idea that you are not a math person, this video might change your point of view! “Brains Grow and Change” is from YouCubed.org, Week 4 of Inspirational Math, Day 5: <https://bhi61nm2cr3mkdgk1dtaov18-wpengine.netdna-ssl.com/wp-content/uploads/2015/06/Brains-Grow-Change.mp4>.

### NumberTalk: 43 – 28

Number talks help you realize how many different, correct, ways there are to solve the same problem. They help you develop flexibility with numbers. Eventually, we will look for generalizations, extending our number talks into algebra. Consider the following problem:

I had $43 with me when I went shopping last week.

I spent $28 on fruit. How much do I have left?

Solve the problem mentally. Find at least one other way to solve the problem mentally.   
When asked to do so, share your solution and one of the ways you solved the problem.

When everyone is ready, you will be asked to share your solution. Once all solutions are expressed, you will take turns sharing solution processes you found. When sharing your solution process, answer the questions below:

* Answer: Which solution are you defending?
* Ask: Does anyone have a question for me?
* Ask: Does anyone have another way to solve the problem?

Remember, your brain grows when you make mistakes and even more when you resolve them.

### Activity: Analyzing the 100s Chart

Generalization is an important concept in algebra. It is the basis of the use of variables. Think about what the word, variable, means when you are not thinking about mathematics. Share your ideas about this.

We will continue to refine our meaning of variable through today’s activities. Work on the student page, *Analyzing the 100s Chart*. As with the *Four Fours* collection of activities, use your creativity. Work to give clear descriptions of any patterns you find. Color code each pattern and illustrate it in the same color on the 100s Chart.

Analyzing the 100s Chart

1. Study the 100s Chart.

a. What must be added to a number to get the number immediately to its right?

b. What must be added to a number to get the number directly below the number?

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| 100 | 101 | 102 | 103 | 104 | 105 | 106 | 107 | 108 | 109 |
| 90 | 91 | 92 | 93 | 94 | 95 | 96 | 97 | 98 | 99 |
| 80 | 81 | 82 | 83 | 84 | 85 | 86 | 87 | 88 | 89 |
| 70 | 71 | 72 | 73 | 74 | 75 | 76 | 77 | 78 | 79 |
| 60 | 61 | 62 | 63 | 64 | 65 | 66 | 67 | 68 | 69 |
| 50 | 51 | 52 | 53 | 54 | 55 | 56 | 57 | 58 | 59 |
| 40 | 41 | 42 | 43 | 44 | 45 | 46 | 47 | 48 | 49 |
| 30 | 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 |
| 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 |
| 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |

2. Replace 6 with the letter *n* so that *n* = 6. When *n* = 6, notice that 7 = *n* + 1. Rewrite 8 and 5 in terms of *n* where *n* = 6. Also write the expressions in the 100s chart above.

3. Replace the number 23 with the letter *m* so that *m* = 23. What are the next two odd numbers after 23? Write these odd numbers using the letter *m* instead of 23. Write the expressions in the 100s chart.

4. How can you represent the number just below 23 on the 100s chart in terms of *m* where   
*m* = 23? How can you represent the number just above 23 on the 100s chart using *m*? Write the expressions in the 100s chart.

|  |  |  |  |
| --- | --- | --- | --- |
| 5. Suppose the rectangle at right is located somewhere in  the 100s chart. Write each value in the rectangle in  terms of *x*.  6. Notice that every rectangle in the 100s chart with the same dimensions as the rectangle at right can be labeled exactly the same way. The initial value of *x* is irrelevant. Why is that? |  |  |  |
|  | *x* |  |
|  |  |  |
|  |  |  |

7. What patterns do you see in the 100s chart? Find at least 3. Color code each pattern in the chart. Describe each pattern in writing. Include a description of how to move to successive numbers and how much is added or subtracted each time you make the move. Explain why the pattern works as you say it does.

For the student page, *Analyzing the 100s Chart*, share your solutions to each problem with your group. Share two of the more challenging patterns your group found with the class.

Of the patterns shared by the class, choose 2 to work on with your group. Together, describe the pattern carefully then figure out why each pattern works. Does the starting number matter? Be ready to share with the class.

### 1.6 Homework

1. Go back to the first section of Lesson 1.1, *How to Learn Math for Students*. Complete problem 6 on page 5. You will hand in your written reflections for problem 6 in class.

2. Revisit problem 6 in the student page, *Analyzing the 100s Chart*, on page 32. Because the initial value of *x* is irrelevant, *x* is considered a variable. Once a particular value of *x* is chosen, then all of the values in the rectangle that are currently represented by expressions in terms of *x* can be determined. In this second case, *x* is a placeholder. Look at the expressions and equations below. For each, decide if *x* is a variable or if *x* is a placeholder.

a. *x* + 1 = 3 b. *x* + 1 = *y* c. 2*x* = *x* + 1d. 2*x* – 1 = *y*

3. You probably have learned to combine like terms in previous mathematics classes. As with the Order of Operations, there are mathematical reasons underlying which terms can be combined.

When children learn to multiply, they use arrays such as those at right to make sense of repeated addition. The product of the dimensions of an array is the number of squares in the array; the number of rows shows how many times the number of squares in the first row is repeated. The student page, *Combining Like Terms*, builds on this experience in learning multiplication to help you think about the meaning of the terms 1, *x*, and *x*2 in the context of area as you model expressions with algebra tiles. Complete the student page. Write down any questions you have as you work so you can ask them in class.

4. There are free apps online to help you make sense of combining like terms and adding polynomials using algebra tiles. Use the apps on this website, <http://media.mivu.org/mvu_pd/a4a/homework/index.html>. Read the introductory pages carefully. These pages tell you how to use the apps. Navigate through the introduction using the arrow in the upper right corner. Watch the videos and play with the apps for *Simplifying Expressions and Adding Polynomials*. Solve at least 2 problems at each difficulty level, until you conceptually understand the link between the algebra tiles and the expressions they represent. This work should take at most 20 minutes.



Combining Like Terms

1. Find the area of each of the algebra tiles shown. The side lengths are labeled. The variable, *x*,represents an unknown length.

2. Translate each of the figures into algebra. Explain your translation. Simplify each expression as much as possible and explain why you cannot simplify further.

a.

b.

c.

d.

3. Draw algebra tiles to illustrate each expression. Use your illustration to simplify each of the algebraic expressions. Explain how you know you are correct.

|  |
| --- |
| a. 2*x* + 4 + 3*x* + 7 = |
| b. 4*x* + 8 – 3*x* + 2 = |
| c. 3*x* + 8 – (4*x* + 2) = |
| d. 7*x* – 2(3*x* – 1) + *x* = |

4. Summarize your work with combining like terms. Which terms can you combine? How do you know?

5. Simplify the following expressions. Show and explain your work.

a. 3 + 5*x* – *x* + 7 = b. 3 + 5*x* – (*x* + 7) =

c. 4 + 2(*x* – 1) = d. 15*x* – 4(2 – 3*x*) + 7 =

## 1.7. More with Patterns and Variables: Rectangular Patterns in the 100s Chart

Throughout this chapter, you have experienced how problems can be approached in a variety of ways. For example, to solve arithmetic problems such as 43 – 28, students found at least 8 ways to do this. Using four fours and four numbers, you found many ways to express most of the numbers 1 through 20. You found many patterns in the 100s chart. By now, we hope you are finding that mathematics is a way of thinking and communicating that can be very creative. One of the hallmarks of mathematics is sense-making in your own way, not in a way prescribed for you. In this lesson you will use your creativity based in sense making to find more ways to simplify an arithmetic expression and look for more patterns in the 100s Chart.

### Number Talk: 93 – 27

Solve the following problem in as many ways as you can.

Mentally solve 93 – 27 in as many ways as you can.

Raise a thumb when you have a solution.   
Raise a finger for each additional solution process you find.

One way a student solved 93 – 27 was by adding 3 to both 93 and 27 to get 96 – 30 = 66. The student also solved the problem by adding 4 to both numbers to get 97 – 31 = 66. Does this strategy always work? Why or why not? Is the student’s process related to the 100s chart? Find the numbers on the 100s chart and explain what’s happening to the numbers. Use algebra to explain why this process works.

### Activity: Rectangle Patterns in the 100s Chart

In Analyzing the 100s Chart, you found patterns in rows, columns, or diagonals. This time your pattern search will be confined to rectangles.

|  |  |  |
| --- | --- | --- |
| 84 | 85 | 86 |
| 74 | 75 | 76 |
| 64 | 65 | 66 |
| 54 | 55 | 56 |

Consider the rectangle at right. What patterns do you see in the numbers in this rectangle?

Let *x* = 84. In the rectangle, write the other values in the rectangle in terms of *x*.

* Use your relabeling to justify any numeric patterns you found.
* Find another pattern in the relabeled rectangle.
* Does your pattern work for a 4 by 3 rectangle with upper left corner labeled 51?
* Does your pattern work for any other 4 by 3 rectangles? Why or why not?

Work on the student page alone for at least 10 minutes. When each member of your group has found at least one rectangle pattern, share the patterns you found with your group.

As a group, choose your 3 favorite patterns, including at least one that you think other groups won’t find. Share your patterns with the class. As a group, choose 2 patterns shared by other groups to justify. Use variables and algebra to justify that each pattern works.

Rectangle Patterns in the 100s Chart

1. Using a different color to outline each rectangle, draw three rectangles following the lines in the 100s chart with the following rules:

a. Make sure all three rectangles have different dimensions.

b. Make one rectangle large and one rectangle small.

c. Make only one square.

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| 100 | 101 | 102 | 103 | 104 | 105 | 106 | 107 | 108 | 109 |
| 90 | 91 | 92 | 93 | 94 | 95 | 96 | 97 | 98 | 99 |
| 80 | 81 | 82 | 83 | 84 | 85 | 86 | 87 | 88 | 89 |
| 70 | 71 | 72 | 73 | 74 | 75 | 76 | 77 | 78 | 79 |
| 60 | 61 | 62 | 63 | 64 | 65 | 66 | 67 | 68 | 69 |
| 50 | 51 | 52 | 53 | 54 | 55 | 56 | 57 | 58 | 59 |
| 40 | 41 | 42 | 43 | 44 | 45 | 46 | 47 | 48 | 49 |
| 30 | 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 |
| 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 |
| 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |

2. What patterns do you notice among the numbers in one of the rectangles? Find at least three patterns. Each pattern you find must not extend beyond the rectangle.

3. Of the patterns you found, do any work for all three rectangles? Explain.

4. Choose one of the patterns you found. Explain why the pattern works.

5. Challenge yourself to use variables to show why the pattern works.

### 1.7 Homework

1. Go back to the first section of Lesson 1.1, *How to Learn Math for Students*. Complete problem 7 on page 5. You will hand in your written reflections for problem 7 in class.

2. Work on the *Rectangle* *Patterns in the 100s Chart* student page. For a pattern that has not been justified in class, explain why the pattern works. Challenge yourself to use variables to generalize the pattern so that it works regardless of where you draw the rectangle in the 100s chart.

3. For an extra challenge, find and work on a pattern that interests you from the *Rectangle* *Patterns in the 100s Chart* student page. Justify it. Translate the pattern into algebra and show that it always works.

## 1.8 Patterns and Variables: Justifying Patterns with Algebra

As in Lesson 1.8, some of our number talks are related to our work with the 100s Chart. Solve the next problem in as many ways as you can. Think about how it is related to the 100s Chart.

### Number Talk: 96 + 29

Mentally solve 96 + 29 in as many ways as you can.

Raise a thumb when you have a solution.   
Raise a finger for each additional solution process you find.

* How does this number talk relate to your work with patterns in the 100s Chart?
* Will your process work for adding any two numbers? Why or why not?

### Activity: Sharing Rectangle Patterns in the 100s Chart

Share your justifications of patterns you found for homework with your group. As a group, decide which two patterns to justify in words and algebraically for the class. For your algebraic justification, label your rectangle as generally as you can, with *x* as the upper left corner of the rectangle and the other numbers in the rectangle based on *x*.

### 1.8 Homework

1. Go back to the first section of Lesson 1.1, *How to Learn Math for Students*. Complete problem 8 on pp. 5 and 6. You will hand in your written reflections for problem 8 in class.

2. a. Draw a 3 by 3 box around 9 numbers in the October 2020 calendar. An example is shown. Use the example or draw your own 3 by 3 box.

b. Find two different patterns among the numbers in the 3 by 3 box. Describe each pattern carefully.

c. Use algebra to justify one of the patterns you found in problem 2b. Indicate which pattern you are justifying.

3. A famous pattern called Pascal’s Triangle is shown in the figure.

a. Study the pattern. How are the numbers from one row related to the numbers in the next row?

b. Fill-in the missing numbers in the last row. Extend the pattern at least one additional row.

c. Describe 2 patterns that you see in this arrangement of numbers.

4. a. Using only mental arithmetic, solve each problem in the table below. Record how you thought about the numbers to help you add them mentally.

|  |  |  |
| --- | --- | --- |
| **Simplify** | **Solution** | **How did you solve the problem?** |
| 25 + 26 |  |  |
| 15 + 16 |  |  |
| 39 + 40 |  |  |
| 18 + 19 |  |  |
| 69 + 70 |  |  |
| 163 + 164 |  |  |

b. Write the sum 25 + 26 in terms of *x* with *x* = 25. Show the sum in terms of *x*.

c. What other problems did you solve using the same strategy you used to solve 25 + 26? Color code the problems that you solve using the same strategy.

d. Did you use the same strategy to mentally solve all of the problems? If you used more than one strategy, color code problems you solved using the same strategy. Do this for each strategy you used. Describe each strategy.

## 1.9 Algebra and Number Puzzles

We have made introductions and studied its underlying mathematics. We looked at the order of operations and saw that the order in which we complete operations is purposeful rather than just a convention to memorize. We’ve experienced finding patterns in the 100s Chart and justifying them with algebra. We’ve seen mathematics arise in unlikely places. This lesson expands our repertoire of unlikely places for mathematics, magic number puzzles. Could it be that math really is everywhere?

Throughout human history of the past 2000 years, humans have been fascinated by puzzles. In this lesson, we play with some of them and see how they are related to algebra.

### Activity: Number Puzzles and Algebra

Consider these two magic number puzzles.

**Number Puzzle 1:**Choose a whole number between 1 and 10.   
Double it.   
Add 4 to the result.   
Divide the result by 2.   
Subtract the number you started with.   
What number do you get?

How does this puzzle work? Act out each step of the puzzle using objects (algebra tiles or blocks and beans will work!) Let one type of object represent *x* and another type of object represent units. Finally, solve the puzzle using a variable to represent an unknown number and algebra to write then simplify expressions.

**Number Puzzle 2**:   
Choose a number.   
Multiply it by 6.   
Add 12.   
Divide the result by 2.   
Add 9.   
Divide the result by 3.   
What is your number? How does it compare with your starting number?

Determine a solution. Figure out how this puzzle works, first with materials, then with algebra.

Discuss the solutions to both puzzles, including justification with materials and with algebra and the relationship between these.

### Activity: Translating Magic Number Puzzles to Algebra to Justify Them

The student page, *Translating Magic Number Puzzles to Algebra to Justify Them*, includes several magic number puzzles for you to solve with your group. Follow the directions on the student page. Be ready to discuss your solutions with the class.

Translating Magic Number Puzzles to Algebra to Justify Them

**Directions**: Try each of the Magic Number Puzzles. Figure out how each one works. Write an informal justification for each puzzle. For one of the puzzles, show why it works using algebra.

1. a. Consider this puzzle:

I. Pick a number.

II. Multiply the number by 2.

III. Add 10 to the total.

IV. Divide the total by 2.

V. Subtract the first number you chose from the result in problem 2d.

b. What number do you get? Why?

2. a. Consider this puzzle:

I. Write down your shoe size rounded to a whole number.

II. Multiply the number by 5

III. Add 50.

IV. Multiply the result by 20.

V. Add this year.

VI. Subtract 1000.

VII. Subtract the year you were born.

VIII. What is special about the number you get?

b. Why does this puzzle work?

c. Does it work for every shoe size? If not, for what shoe sizes do you have to adjust the puzzle to make it work? How do you have to adjust the puzzle?

3. a. Consider this puzzle:

I. Write your age on a slip of paper.

II. Add 90 to your number.

III. Cross out the leftmost digit from the result.

IV. Add the digit you crossed off to the result.

V. Add 9 to the result.

VI. What is the final result? How does it relate to the number you chose?

b. Are there any ages for which this puzzle doesn’t work? How do you know?

4. a. Consider this puzzle:

I. Write the year of your birth.

II. Double it.

III. Add 5.

IV. Multiply the result by 50.

V. Add your age.

VI. Add 365.

VII. Subtract 615.

VIII. What is the final result? How does it relate to the numbers you chose in the problem?

### 1.9 Homework

5. There were 100 chocolates in a box. The box was passed from person to person in one row. The first person took one chocolate. Each person down the row took one more chocolate than the person before. The box was passed until it was empty. What is the largest number of people that could have removed chocolates from the box? How do you know?

6. Consider this puzzle:

I. Choose any number.

II. Multiply the number by 100.

III. Subtract the original number.

IV. Add the digits in your answer.

a. Try steps I–IV using single digit numbers (1 through 9). What numbers are possible? How do you know?

b. Try steps I–IV using numbers from 10 through 99. What numbers are possible? How do you know?

c. What numbers are possible if you complete the puzzle using 3-digit numbers? Show the numbers you used to get each answer to step IV.

d. One student found that the answer for step IV could be 18, 27, or 36.

i. How many digits must the original number have for the puzzle to give 27 as an answer? Find a number that works.

ii. How many digits must the original number have for the puzzle to give 36 as an answer? Find a number that works.

iii. Are any other numbers possible for the answer to step IV? For any additional step IV solutions you find, show the number you used to get the answer.

1. Go back to the first section of Lesson 1.1, *How to Learn Math for Students*. Complete problem 9 on page 6. You will hand in your written reflections for problem 9 in class.

2. Finish solving the student page, *Translating Magic Number Puzzles to Algebra to Justify Them*. Be ready to discuss solutions and ask any remaining questions in class.

3. National Public Radio regularly congratulates patrons celebrating birthdays and anniversaries. Here’s an announcement made in 2017:

*Congratulations to Laurie on her 24th birthday, counting backwards since 2002.*

a. What was Laurie’s actual age on the day of the announcement?

b. How old will Laurie be in 2030?

c. How old will Laurie actually be when she says she is 1-year old (assuming she keeps this counting scheme going)? Is she likely to live that long?

d. Write an expression that will help you determine Laurie’s NPR age in any year after 2002. Say how you know your expression is correct.

4. Study the Magic Number Puzzle. Try it out for a few numbers, then show how and why it works using Algebra or pictures.

I. Choose a number.

II. Multiply by 6.

III. Add 12.

IV. Divide by 2.

V. Subtract 6.

VI. Divide by 3.

VII. What is special about the number you get?

b. Why does this puzzle work?

5. Find a magic number puzzle different from those you encountered in this lesson. Write down or print out a puzzle you find. Solve the puzzle and figure out how it works. Be ready to share with your group.